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# THE GALERKIN FINITE ELEMENT METHOD FOR 1D BOUNDARY VALUE PROBLEMS

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**Abstract.**
This article presents an algorithm for solving 1D boundary value problems using the Galerkin finite element method. Furthermore, new local basis functions for the Galerkin finite element method are introduced, and it is established that these functions are fundamental splines.

**Keywords.**
Basis functions, ordinary differential equation, boundary value problem, finite element, interpolation, interpolation formula, bilinear form.

The concept of a boundary value problem for ordinary differential equations of the second order can be stated in general as follows. We consider the second order differential equetion

$$Lu \equiv -\frac{d}{dx}\left(p\frac{du}{dx}\right)+qu=f(x), a\le x\le b \tag{1}$$

with the boundary conditions

$$\alpha_1 u(a)+\beta_1 u'(a)=\gamma_1, \alpha_2 u(b)+\beta_2 u'(b)=\gamma_2. \tag{2}$$

The differential equation (1) with the boundary conditions (2) is called a boundary value problem. Here $p\in C^1[a,b], q, f\in C[a,b]$ and $p(x)\ge k>0, q(x)\ge 0$ for $x\in[a,b], k=const, \alpha_i, \beta_i, \gamma_i (i=1,2)$ are given numbers.

Now we deal with the approximate solution of the boundary value problem (1)-(2) using the finite element method. We integrate (1) over the interval $[a,b]$ multiplying by an arbitrary function $v\in C^1[a,b]$ satisfying the boundary conditions (2) and using the formula of integration by parts, we get

$$\int_a^b (pu'v'+quv)dx=\int_a^b fvdx. \tag{3}$$

It should be noted that equality (3) is in some sense equivalent to boundary value problem (1)-(2). Indeed, if the function u is a solution of the boundary value problem (1)-(2), then the function u satisfies equality (3) and vice versa.

We use equation (3) to approximate the boundary value problem (1)-(2).

Let us consider, for example, the Galerkin method. Let us given linear independent functions $\xi_0,\xi_1,...,\xi_n\in C^1[a,b]$ satisfying the boundary conditions (2). In that case, the approximate solution of the boundary value problem (1)-(2) is sought in the following form:

$$u_n(x)=\sum_{j=0}^{n} c_j\xi_j(x). \tag{4}$$

Since the function $u_n(x)$ determined by equality (4) is an approximate solution of the boundary value problem (1)-(2), we have a system of linear equations

$$\int_a^b (pu_n'\xi_i'+qu_n\xi_i)dx=\int_a^b f\xi_i dx, i=0,1,...,n \tag{5}$$

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Taking into account (4), we write this system of linear equations in the following form

$$\sum_{j=0}^{n} a_{ij}c_j = b_i, i = 0,1,\dots,n$$

or

$$Ac = b,$$

where

$$A = \left(a_{ij}\right)_{i,j=0}^{n}, c = (c_0,\dots,c_n)^T, b = (b_0,\dots,b_n)^T,$$

$$a_{ij} = a(\xi_i,\xi_j) = \int_a^b \left(p\xi_i'\xi_j' + q\xi_i\xi_j\right)dx,\ b_i = b(\xi_i) = \int_a^b f\xi_i dx,$$

and $\xi_i(x)$ $(i = 0,1,\dots,n)$ was created [1, 2]:

$$\xi_0(x) = \begin{cases} \dfrac{\sin(\omega x - \omega x_1)}{\sin(\omega x_0 - \omega x_1)}, x_0 \le x < x_1, \\ 0, x_1 \le x \le 1, \end{cases} \quad (6)$$

$$\xi_i(x) = \begin{cases} 0, x_0 \le x < x_{i-1}, \\ \dfrac{\sin(\omega x - \omega x_{i-1})}{\sin(\omega x_i - \omega x_{i-1})}, x_{i-1} \le x < x_i, \\ \dfrac{\sin(\omega x - \omega x_{i+1})}{\sin(\omega x_i - \omega x_{i+1})}, x_i \le x < x_{i+1}, \\ 0, x_{i+1} \le x \le 1, \end{cases} \quad (i = 1,2,\dots,n-1) \quad (7)$$

$$\xi_n(x) = \begin{cases} 0, x_0 \le x < x_{n-1}, \\ \dfrac{\sin(\omega x - \omega x_{n-1})}{\sin(\omega x_n - \omega x_{n-1})}, x_{n-1} \le x \le x_n. \end{cases} \quad (8)$$

Here, $0 = x_0 < x_1 < \dots < x_n = 1$, $x_i = ih$, $h = \dfrac{1}{n}$, $i = 0,1,\dots,n$ and $\omega \in \mathbb{R}\setminus\{0\}$.

Solving the system of linear equations (5), we find the coefficients $c_j, j = \overline{0,N}$ and get the approximate solution $u_n(x)$.

Since the functions $\xi_0,\xi_1,\dots,\xi_n \in C^1[a,b]$ are linear independent, it follows that the symmetric bilinear form $a_{ij}(i,j = \overline{0,n})$ is positive definite. This, in turn, means that the main matrix $A$ of the system of linear equations is positive. Therefore, the solution of the system of linear equations (5) exists and is unique.

**Remark 1.** For the basis functions $\xi_i(x)$ $(i = 0,1,\dots,n)$, defined by the expressions $(6)-(8)$, the relation

$$\xi_i(x_j) = \begin{cases} 1, i = j, \\ 0, \text{otherwise} \end{cases}$$

i.e.,

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$$\xi_i\left(x_j\right)=\delta_{ij}, i=0,1,...,n, j=0,1,...,n,$$

holds, where $\delta_{ij}$ is the Kroniker symbol.

**Remark 2.** The basis functions $\xi_i\left(x\right)$ $\left(i=0,1,...,n\right)$ are local linear trigonometric fundamental splines in a Hilbert space [2].

With a suitable choice of basis functions, the accuracy of the approximation method improves as $n$ increases.

**Conclusion**

This work describes an algorithm for solving 1D boundary value problems using the Galerkin finite element method. New local basis functions are introduced, and it is established that these functions constitute fundamental splines.

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