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THE DEVELOPMENT OF THE THEORY OF ALGEBRAIC EQUATIONS IN THE SIXTEENTH AND SEVENTEENTH CENTURIES: AN ANALYSIS OF THE METHODS OF CARDANO, TARTAGLIA, AND BOMBELLI

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Abstract: This article analyzes the formation and development of the theory of algebraic equations in the sixteenth and seventeenth centuries from a historical and mathematical point of view. The main focus is placed on the problem of solving cubic and quartic equations, particularly on the rise of algebraic thinking to a new stage through the works of Tartaglia, Cardano, Ferrari, and Bombelli. The article discusses the reduction of a cubic equation to its depressed form, Cardano's formula, discriminant cases, the historical necessity of complex numbers, and the mathematical significance of Bombelli's approach. In addition, the methodological importance of the history of algebraic solutions in higher mathematics education is demonstrated. In the practical section, several examples related to cubic equations are solved step by step, and a modern interpretation of historical formulas is presented. The results of the study show that the methods for solving algebraic equations developed in the sixteenth and seventeenth centuries had a strong influence not only on the development of algebra, but also on the subsequent progress of mathematical analysis, number theory, and the theory of complex numbers.

Keywords: algebraic equation, cubic equation, Cardano's formula, Tartaglia's method, Bombelli, Ferrari, complex numbers, history of mathematics, mathematics of the sixteenth and seventeenth centuries.

1. Introduction

In the history of mathematics, the problem of solving algebraic equations occupies a special place. Although the mathematicians of ancient Babylon, Greece, India, and the Arab world developed methods for solving first- and second-degree equations, the problem of solving third- and fourth-degree equations in a general form remained open for a long time. In particular, the solution of cubic equations was regarded as one of the most difficult problems in the mathematics of the late Middle Ages and the Renaissance.

One of the important features of sixteenth-century European mathematics was that algebra gradually began to transform from a practical art of computation into a theoretical science. During this period, the needs of trade, geodesy, astronomy, mechanics, and engineering required the improvement of mathematical methods. The problem of solving equations was directly connected not only with practical calculations but also with the development of abstract mathematical thinking.

Several famous names are associated with the history of solving cubic equations: Scipione del Ferro, Niccolò Tartaglia, Gerolamo Cardano, Lodovico Ferrari, and Rafael Bombelli. The works of these scholars are closely interconnected, and together they opened a new era in algebra. Cardano published methods for solving cubic and quartic equations in his famous work *Ars Magna*. Tartaglia discovered an important algorithm for solving cubic equations. Ferrari developed a method for solving fourth-degree equations. Bombelli was among the first mathematicians to understand the necessity of working with complex numbers.

The relevance of this article lies in the fact that studying the historical development of the theory of algebraic equations has significant methodological importance in modern higher mathematics education. If students understand Cardano's formula or the theory of complex numbers not merely as ready-made results but as scientific discoveries arising from historical necessity, they can grasp the internal logic of the topic more deeply.

The purpose of this article is to analyze the development of the theory of algebraic equations in the sixteenth and seventeenth centuries on the basis of the works of Cardano, Tartaglia, Ferrari, and Bombelli, and to show the importance of this process in modern mathematics education.

The objectives of the article are as follows: to study the historical formation of the solution of cubic equations; to reveal the mathematical content of Cardano's formula; to explain the essence of Tartaglia's method; to demonstrate the historical necessity of complex numbers through Bombelli's approach; and to justify theoretical results through solved examples related to the topic.

2. Methods

This article uses historical-mathematical analysis, comparison, algebraic modeling, and the modern interpretation of formulas. As the main source, the sections on algebraic equations in G. G. Zeuthen's work devoted to the history of mathematics in the sixteenth and seventeenth centuries were used. In this source, the problem of solving third- and fourth-degree equations, as well as the works of Cardano, Tartaglia, Ferrari, and Bombelli, are presented in historical sequence [1].

The research methodology is based on the following directions.

First, using a historical approach, the stages of development of the theory of algebraic equations in the sixteenth century were examined. In this process, the emergence of mathematical ideas was interpreted not only in the form of formulas, but also in connection with the scientific environment, debates, problems, and practical needs of the period.

Second, through algebraic analysis, the process of reducing a general cubic equation of the form

$$ax^3+bx^2+cx+d=0$$

to the depressed cubic form

$$y^3+py+q=0$$

was explained. Although this transformation seems simple from the point of view of modern algebra, historically it was a very important step.

Third, the mathematical basis of Cardano's formula was shown. For the depressed cubic equation

$$x^3+px+q=0\ ,$$

the solution is expressed by the following formula:

$$x=\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}\ .$$

In this formula, the expression

$$\Delta=\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3$$

plays an important role in determining the nature of the roots of the equation.

Fourth, Bombelli's approach was analyzed in order to explain the historical necessity of complex numbers. In particular, in cubic equations having three real roots, the appearance of the square root of a negative number under Cardano's formula created a serious theoretical problem for sixteenth-century mathematicians.

3. Results

As a result of the analysis, the following scientific and methodological conclusions were obtained.

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First, in the sixteenth century, the problem of solving third-degree equations initiated a qualitatively new stage in the history of algebra. Whereas in earlier periods algebra was considered mainly as a tool for solving numerical problems, in the time of Cardano and Tartaglia it began to take shape as a science of general formulas and abstract relations.

Second, Cardano's formula strengthened the idea of expressing solutions of algebraic equations by radicals. This formula has not lost its historical significance in modern higher algebra, since it clearly shows the relationship between the roots of an equation and its coefficients.

Third, Bombelli's works created an important theoretical foundation for the theory of complex numbers. The necessity of extracting square roots from negative numbers in the solution of cubic equations led to the acceptance of complex numbers not as artificial objects, but as mathematical objects arising from genuine necessity.

Fourth, Ferrari's method for solving fourth-degree equations opened the way to a deeper understanding of algebraic structures. This method demonstrates the idea of solving higher-degree equations by reducing them to auxiliary equations of lower degree, in particular to a cubic equation.

Fifth, the history of algebraic equations can serve as an effective methodological tool in the teaching of higher mathematics. Explaining the historical process through which formulas appeared increases students' interest in the topic, develops their logical thinking, and helps them understand the meaning of mathematical concepts more deeply.

4. Discussion

4.1. Historical Roots of the Problem of Cubic Equations

The problem of solving third-degree equations has attracted mathematicians since ancient times. Babylonian mathematicians solved certain special equations by numerical methods. Greek mathematicians, on the other hand, attached great importance to geometric methods. For example, some cubic problems could be solved with the help of conic sections. However, there was no general algebraic formula.

During the period of Arabic mathematics, al-Khwarizmi, Omar Khayyam, and other scholars made major contributions to the classification of equations. Omar Khayyam attempted to solve cubic equations by geometric methods. Nevertheless, an algebraic formula in radicals had not yet been found. Thus, by the sixteenth century, the cubic equation problem existed in the history of mathematics as a mature but still unresolved problem.

During the Renaissance, scientific competitions related to algebraic equations were widespread in Italy. Mathematicians posed difficult problems to one another and demonstrated their skill by solving them. It was precisely this scientific environment that led to the emergence of the works of Tartaglia and Cardano.

4.2. The Essence of Tartaglia's Method

Tartaglia found a method for solving certain types of third-degree equations. In modern notation, his approach corresponds to the depressed cubic equation

$$x^3 + px + q = 0.$$

To solve this equation, the substitution $x = u + v$ is introduced. Then

$$(u+v)^3 + p(u+v) + q = 0 \ .$$

Expanding, we obtain

$$u^3 + v^3 + 3uv(u+v) + p(u+v) + q = 0 \ .$$

If the condition

$$3uv + p = 0$$

is satisfied, the equation is simplified as follows:

$$u^3 + v^3 + q = 0 \ .$$

Therefore, $u^3 + v^3 = -q$, and $uv = -\frac{p}{3}$.

From this we obtain

$$u^3v^3 = \left(-\frac{p}{3}\right)^3.$$

If u^3 and v^3 are regarded as unknowns, they can be found as two numbers whose sum and product are known. As a result, Cardano's formula follows.

The significance of Tartaglia's method lies in the fact that it made it possible to solve a cubic equation through two auxiliary quantities. This method was later generalized and published by Cardano.

4.3. The Mathematical Meaning of Cardano's Formula

Cardano's formula for the depressed cubic equation

$$x^3 + px + q = 0$$

has the following form:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}},$$

Where

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3.$$

The expression Δ in this formula determines the nature of the roots of the equation. If $\Delta > 0$, the equation has one real root and two complex conjugate roots. If $\Delta = 0$, at least two of the roots are equal. If $\Delta < 0$, the equation has three real roots, but complex numbers appear in the calculation process when Cardano's formula is used.

The case $\Delta < 0$ is especially important from a historical point of view. In this case, although the final roots are real, square roots of negative numbers appear in the intermediate calculations. This led to the necessity of recognizing complex numbers as a natural part of algebraic computation.

4.4. Bombelli and the Historical Necessity of Complex Numbers

Bombelli was one of the mathematicians who tried to work consistently with the "imaginary" quantities that appeared in Cardano's formula. His approach showed that if expressions such as $\sqrt{-1}$ arise in the calculation process, they should not be rejected completely; rather, they can be treated according to special algebraic rules.

For example, the equation

$$x^3 = 15x + 4 \text{ or } x^3 - 15x - 4 = 0$$

has a real root. Indeed, substituting $x = 4$ gives

$$4^3 - 15 \cdot 4 - 4 = 64 - 60 - 4 = 0.$$

However, solving it by Cardano's formula gives the expression

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

Although this expression appears complex, its final value is the real number 4. Bombelli analyzed precisely such cases and made an important step toward understanding the rules for working with complex numbers.

This phenomenon has a deep philosophical and mathematical meaning in the history of algebra. Mathematical objects sometimes initially seem "imaginary," "artificial," or "meaningless," but later become an inseparable part of a theory. Complex numbers are a clear example of this process.

4.5. Ferrari's Method and Fourth-Degree Equations

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The problem of solving fourth-degree equations is also considered one of the major achievements of sixteenth-century mathematics. Ferrari showed that a fourth-degree equation could be reduced to a cubic auxiliary equation by means of suitable transformations.

The general fourth-degree equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

is first transformed by the substitution $x = y - \frac{b}{4a}$, which eliminates the cubic term and gives the form

$$y^4 + py^2 + qy + r = 0 \ .$$

The main idea of Ferrari's method is to transform the left-hand side into a complete square: $y^4 + py^2 + qy + r = (y^2 + m)^2$ -another expression.

With an appropriate choice of m, the remaining expression also becomes a square, and the equation decomposes into two quadratic equations. To find m, a cubic auxiliary equation is obtained. Therefore, solving a fourth-degree equation is connected with solving a cubic equation.

Ferrari's method is important in the history of algebra because it is an effective example of the idea of reducing higher-degree equations to auxiliary equations of lower degree.

5. Practical Examples

Example 1. Solving a Depressed Cubic Equation by Cardano's Formula

Solve the equation:

$$x^3 - 6x - 9 = 0 \ .$$

Here $p = -6, \quad q = -9$.

We compute the discriminant in Cardano's formula:

$$\Delta = \frac{q}{2}^{2} + \frac{p}{3}^{3} \ .$$

Thus,

$$\Delta = -\frac{9}{2}^{2} + (-2)^3 = \frac{81}{4} - 8 = \frac{81-32}{4} = \frac{49}{4} \ .$$

$$\sqrt{\Delta} = \sqrt{\frac{49}{4}} = \frac{7}{2} \ .$$

Now the solution is

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}} \ .$$

$$-\frac{q}{2} = \frac{9}{2} \ .$$

Therefore,

$$x = \sqrt[3]{\frac{9}{2} + \frac{7}{2}} + \sqrt[3]{\frac{9}{2} - \frac{7}{2}} = \sqrt[3]{8} + \sqrt[3]{1} = 2 + 1 = 3 \ .$$

Checking:

$$3^3 - 6 \ 3 - 9 = 27 - 18 - 9 = 0 .$$

Hence, one real root of the equation is

$$x = 3 .$$

In this example, $\Delta > 0$, so the equation has one real root and two complex conjugate roots.

Example 2. Reducing a General Cubic Equation to Depressed Form

Solve the equation:

$$x^3 - 6x^2 + 11x - 6 = 0 .$$

This equation contains a quadratic term. To eliminate it, we introduce the substitution

$x = y + 2$.

In the general case, for the equation $x^3 + ax^2 + bx + c = 0$, one takes $x = y - \frac{a}{3}$. Here $a = -6$, hence $x = y + 2$.

Substituting into the equation gives

$(y+2)^3 - 6(y+2)^2 + 11(y+2) - 6 = 0$.

Expanding, we get

$y^3 + 6y^2 + 12y + 8 - 6(y^2 + 4y + 4) + 11y + 22 - 6 = 0$.

$y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 11y + 22 - 6 = 0$.

Combining like terms: $y^3 - y + 0 = 0$.

Therefore, $y^3 - y = 0 \qquad y(y^2 - 1) = 0 \qquad y(y-1)(y+1) = 0$

Hence, $y_1 = 0, \quad y_2 = 1, \quad y_3 = -1$.

Returning to $x = y + 2$, we obtain

$x_1 = 2, \quad x_2 = 3, \quad x_3 = 1$.

Therefore, $x_1 = 1, \quad x_2 = 2, \quad x_3 = 3$.

This example shows the simplicity and effectiveness of reducing a cubic equation to depressed form.

Example 3. Bombelli's Case: A Real Root Obtained Through Complex Expressions

Solve the equation: $x^3 - 15x - 4 = 0$, here $p = -15, \quad q = -4$.

The discriminant is

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3.$$

$\Delta = (-2)^2 + (-5)^3 = 4 - 125 = -121$.

Thus, $\sqrt{\Delta} = \sqrt{-121} = 11i$.

By Cardano's formula,

$x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$.

Now we use the idea of Bombelli's method. Suppose that

$\sqrt[3]{2 + 11i} = a + bi$ and $\sqrt[3]{2 - 11i} = a - bi$.

Their sum is $2a$. We want to find a real root. It is known that

$(2+i)^3 = 8 + 12i + 6i^2 + i^3 = 8 + 12i - 6 - i = 2 + 11i$.

Therefore, $\sqrt[3]{2 + 11i} = 2 + i$.

Similarly, $\sqrt[3]{2 - 11i} = 2 - i$.

Hence, $x = (2+i) + (2-i) = 4$.

Checking: $4^3 - 15 \ 4 - 4 = 64 - 60 - 4 = 0$.

Thus, $x = 4$.

The historical significance of this example is very great: the real root of the equation is obtained through complex intermediate expressions. This case clearly demonstrates the necessity of introducing complex numbers into algebraic computation.

Example 4. Solving Another Cubic Equation by Cardano's Formula

Solve the equation: $x^3 + 3x - 4 = 0$.

Here $p = 3, \quad q = -4$.

The discriminant is

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$$\Delta = \frac{-4}{2}^{2} + \frac{3}{3}^{3} = (-2)^2 + 1^3 = 4 + 1 = 5 .$$

Cardano's formula gives

$$x = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} .$$

This expression gives a real root. By direct verification, we see that $x = 1$ is a root:

$$1^3 + 3 \ 1 - 4 = 1 + 3 - 4 = 0 .$$

Therefore, $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} = 1$.

We factor the equation: $x^3 + 3x - 4 = (x-1)(x^2 + x + 4)$.

Solving the quadratic equation,

$$x^2 + x + 4 = 0 ,$$

we find

$$D = 1 - 16 = -15 .$$

$$x = \frac{-1 \pm \sqrt{-15}}{2} = \frac{-1 \pm i\sqrt{15}}{2} .$$

Thus, all roots are

$$x_1 = 1, \qquad x_2 = \frac{-1 + i\sqrt{15}}{2}, \qquad x_3 = \frac{-1 - i\sqrt{15}}{2} .$$

This example shows that in the case $\Delta > 0$, a cubic equation has one real root and two complex conjugate roots.

6. Methodological Importance in Higher Mathematics Education

Studying the history of algebraic equations is useful in higher mathematics education in several respects. First, this topic familiarizes students with the historical origin of mathematical formulas. Very often, students accept Cardano's formula or the rules for working with complex numbers as ready-made results. A historical approach, however, explains the problems through which these formulas emerged.

Second, the topic of algebraic equations develops abstract thinking in students. For example, the introduction of the substitution $x = u + v$ is not merely a technical procedure; it is the result of creative thinking aimed at simplifying the structure of a complex equation.

Third, Bombelli's example explains the necessity of complex numbers. A student sees that the symbol $i = \sqrt{-1}$ is not simply an artificially introduced concept, but is sometimes necessary even in the process of finding real roots.

Fourth, the historical approach strengthens interdisciplinary connections. The theory of algebraic equations can be studied in connection with the history of mathematics, philosophy, logic, mechanics, and astronomy. This allows students to understand mathematics as a broad scientific and cultural phenomenon.

Fifth, this topic is suitable for problem-based teaching methods. The teacher can pose the following problem questions to students: "Why is the substitution $x = u + v$ useful in a cubic equation?", "Why does a complex number appear when finding a real root?", "Why can a fourth-degree equation be reduced to a cubic auxiliary equation?" Such questions encourage students to think actively.

7. Conclusion

The development of the theory of algebraic equations in the sixteenth and seventeenth centuries is one of the most important stages in the history of mathematics. During this period, general algebraic methods for solving third- and fourth-degree equations were created. Tartaglia found an important method for solving cubic equations, Cardano generalized and published this method, Ferrari developed a method for solving fourth-degree equations, and Bombelli demonstrated the theoretical necessity of working with complex numbers.

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Cardano's formula brought about a fundamental turning point in the theory of algebraic equations. It gave a precise form to the idea of expressing the roots of an equation through its coefficients. However, computations connected with this formula brought the problem of complex numbers to the surface. Bombelli understood this problem deeply and showed that formal algebraic operations could be performed with complex expressions.

Ferrari's method for solving fourth-degree equations demonstrated the idea of solving higher-degree equations through auxiliary equations. This created a foundation for the further development of the theory of algebraic equations.

This historical process is also highly significant for modern higher mathematics education. Studying the history of algebraic equations familiarizes students with the origin, developmental logic, and scientific necessity of mathematical concepts. This strengthens their theoretical knowledge, develops creative thinking, and enhances their mathematical culture.

In general, the works of Cardano, Tartaglia, Ferrari, and Bombelli created a solid foundation not only for the mathematics of their own time but also for the entire subsequent development of mathematics. Their achievements in the field of algebraic equations served as an important historical factor in the development of mathematical analysis, the theory of complex numbers, algebra, and number theory.

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