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SOME THEOREMS FOR RANDOM SUMS OF RANDOM VARIABLES

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Abstract. In the paper the problem of existence of the expectations of functions of random sums is considered. Using total expectation formula the existence of the expectation of function of random sum is proved, in the case when summands do not depend on the index of summation.

Keywords: Random sums, nonnegative random variables, branching presses.

Let a sequence of random variables be defined on some probability space (Ω, F, P)

$$\xi_1, \xi_2, \ldots, \xi_n, \ldots \tag{1}$$

and a non-negative integer random variable ν.

In the theory of summation of random variables, a random sum is a quantity

$$S_\nu = \xi_1 + \xi_2 + \ldots + \xi_\nu ,$$

i.e. the sum of a random number of random variables (see [5]). In the case where the sequence (1) consists of non-negative independent random variables, the random sum S_ν plays a significant role in the theory of branching random processes ([6], [9]), as well as in reliability theory ([2]). For example, for the study of the asymptotic properties of branching random processes (the probabilities of degeneration and continuation of the process, the limit distributions of the sum of the total number of particles, etc.), B.A. Sevastyanov's problem of describing the class of measurable functions $g(x)$ defined on $[0, \ )$ such that from the existence of $Eg(\xi_i)$ $(i \ \ \mathbf{N})$ and $Eg(\nu)$ the existence of mathematical expectation follows $Eg(S_\nu)$.

We first examine the formulated problem for random sums S_ν, where the terms $\xi_1, \ \xi_2, \ldots$ are independent random variables and ν is an independent integer non-negative random variable. According to the formula for the total mathematical expectation, the characteristic function of a random sum S_ν has the following form:

$$\Psi(t) = Ee^{itS_\nu} = \ _{n=0} E(e^{itS_n}; \nu = n) = \ _{n=0} P(\nu = n) Ee^{itS_n} =$$

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$$= \sum_{n=0}^{\infty} P(\nu = n) \prod_{k=1}^{n} f_k(t), \qquad (2)$$

where $S_0 = 0$, $f_k(t) = Ee^{it\xi_k}$ is the characteristic function of random variables. ξ_k. From here on, the entry $E(\xi; A)$ means that the averaging of random variables ξ is performed over a set A, i.e.

$$E(\xi; A) = \int_A \xi(\omega)P(d\omega), \quad A \in F.$$

Taking into account the formulas $ES_\nu^r = i^{-r}\Psi^{(r)}(0)$ from equality (2), we conclude that the existence of a mathematical expectation $max(E\nu^r, E\xi_i^r) < \infty$, $i \in \mathbf{N}$, for an integer , $r > 0$ entails $ES_\nu^r < \infty$, and the value ES_ν^r is a function of the values $E\xi_i^{r_1}$, $E\nu^{r_1}$ for integers r_1 $(0 < r_1 \le r)$. For example, in the case when random variables ξ_i are identically distributed (see [5]),

$$ES_\nu = E\nu E\xi_1, \; ES_\nu^2 = E\nu[E\xi_1^2 - (E\xi_1)^2] + E\nu^2(E\xi_1)^2$$

etc.

From the above it follows that bounded functions and power functions $g(x) = x^r$, $r = 1, 2, \ldots$ belong to the class of measurable functions possessing the properties formulated in the given problem of B.A. Sevastyanov . Below we will establish fairly broad conditions for functions to belong $g(\cdot)$ to the class of functions considered in this problem.

Following B.A. Sevastyanov [6], we introduce the following classes of functions:

1) We will say that a function $g(x)$ defined on $[0, \infty)$ belongs to the class G_1 if $g(x)$ it is non-negative and there exists such $C > 0$ that for any $x, y \ge 0$

$$g(xy) \le Cg(x)g(y). \qquad (3)$$

2) We will say that a function $g(x)$ defined on $[0, \infty)$ belongs to the class G_2 if it is non-negative, non-decreasing and convex.

The following theorem holds:

Theorem 1 [6] . Let the random variables ξ_i of sequence (1) be independent and identically distributed, and let be ν an independent integer non-negative random variable. Then for any function $g \in G_1 \cap G_2$ the inequality holds.

$$Eg(S_\nu) \quad CEg(\xi_1)Eg(\nu). \tag{4}$$

Now we turn to the generalization of Theorem 1 for the case when ξ_i there is a certain dependence between the random variables $F_{k,n} = \sigma(\xi_k,\dots,\xi_n)$ of the sequence (1) and the integer random variable . Let be the $\nu \quad 0$ -algebra σ generated by $n-k+1$ random variables $\xi_k,\dots,\xi_n$.

As is well known [7], a random variable ν is called independent of the future if the event $(\nu \quad n)$ does not depend on $F_{n+1,}$ i.e. equality

$$P((\nu \quad n) \quad A) = P(\nu \quad n)P(A)$$

is satisfied for any $A \quad F_{n+1,}$.

In the book by A.A. Borovkov [1] for the case $g(x) = x$ there is the following statement (Kolmogorov-Prokhorov theorem): Let an integer non-negative random variable ν not depend on the future. Then, if

$$\sum_{k=1} P(\nu \quad k)E|\xi_k| < \quad , \tag{5}$$

That

$$ES_\nu = \sum_{k=1} P(\nu \quad k)E\xi_k. \tag{6}$$

If the random variables are $\xi_k \quad 0$, then condition (5) is redundant. But the non-negativity of the mathematical expectations of the random variables $(E\xi_k \quad 0)$ does not ensure the validity of equality (6) (oral communication by A.A. Borovkov). The latter means that the above Kolmogorov-Prokhorov theorem is unimprovable . It should also be noted that in the case $x = g(x) \quad G_1 \quad G_2$ (with a constant $C = 1$), inequality (4) turns into equality. In fact, if the random variables ξ_i are identically distributed and

$$\nu,\ \xi_1,\ \xi_2, \dots, \xi_n, \dots$$

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- a sequence of independent random variables (not necessarily non-negative), then equality (6) turns into equality

$$ES_{\nu} = E\xi_1 \sum_{n=1}^{\infty} P(\nu \geq n). \tag{7}$$

Now let us prove the following simple statement.

Lemma 1. If N is a random variable with values from $\mathbf{N}$ and $EN < \infty$, then

$$EN = \sum_{n=1}^{\infty} P(N \geq n). \tag{8}$$

Proof of Lemma 1. For $k > 1$ we have

$$\sum_{n=1}^{k} nP(N = n) = \sum_{n=1}^{k} n[P(N \geq n) - P(N \geq n+1)] =$$

$$= \sum_{n=1}^{k} n[P(N > n-1) - P(N > n)] =$$

$$= \sum_{n=1}^{k} (n-1)P(N > n-1) + \sum_{n=1}^{k} P(N > n-1) - \sum_{n=1}^{k} nP(N > n) =$$

$$= \sum_{n=1}^{k} P(N \geq n) - kP(N > k). \tag{9}$$

But

$$kP(N > k) = k \sum_{n=k+1}^{\infty} P(N = n) \leq \frac{k}{k+1} \sum_{n=k+1}^{\infty} nP(N = n).$$

Because

$$EN = \sum_{n=1}^{\infty} nP(N = n) < \infty,$$

That

$$\lim_{k\to\infty} kP(N > k) = 0. \tag{10}$$

Equality (8) follows from relations (9), (10) taking into account that

$$EN = \lim_{k\to\infty} \sum_{n=1}^{k} nP(N = n).$$

The proof of Lemma 1 is complete.

Remark 1. The assertion of the lemma is well known. However, the given proof is not found in the literature on probability theory.

According to the proven lemma, equality (7) can be rewritten as

$$ES_\nu = E\xi_1 E\nu. \qquad (11)$$

The last equality (11) is the famous Wald identity , which plays an important role in the theory of statistical inference.

We now turn to a generalization of the theorem for the case where ν is a random variable independent of the future with respect to the sequence of random variables (1). In some cases, ν it is also called a Wald random variable (see [5]).

Theorem 2. Let be $\xi_1, \xi_2, \dots, \xi_n, \dots$ a sequence of independent non-negative random variables (not necessarily identically distributed), and the random variable ν does not depend on the future. Then for each function $g(x) \quad G_1 \quad G_2$ the inequality holds.

$$Eg(S_\nu) \quad C \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} Eg(\xi_k) \quad g(n) P\{\nu = n\}. \qquad (12)$$

Proof of Theorem 2. According to the formula of total mathematical expectation

$$Eg(S_\nu) = \sum_{n=1}^{\infty} E\left[g(S_\nu); \nu = n\right] = \sum_{n=1}^{\infty} E\left[g(S_n); \nu = n\right].$$

Since the function is $g(\ ) \quad G_1 \quad G_2$, we have

$$Eg(S_\nu) \quad C \sum_{n=1}^{\infty} \frac{g(n)}{n} \sum_{k=1}^{n} E[g(\xi_k); \nu = n] \quad =$$

$$= C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{g(n)}{n} E[g(\xi_k); \nu = n]. \qquad (13)$$

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Changing the order of summation in (13) according to the formula

$$\underset{n=1}{} a_n \overset{n}{\underset{k=1}{}} b_k = \underset{k=1}{} b_k \underset{n=k}{} a_n$$

we get that

$$\underset{n=1}{} \frac{g(n)}{n} \overset{n}{\underset{k=1}{}} E[g(\xi_k);\nu = n] = \underset{k=1}{}\underset{n=k}{} \frac{g(n)}{n} E[g(\xi_k);\nu = n]. \qquad (14)$$

Now we need the following

Lemma 2. Let events A and B $(B \quad A)$ are independent of the event D. Then $A \setminus B$ they D will be independent events.

Proof of Lemma 2. By the conditions of the lemma, the following equalities are satisfied:

$$P\{AD\} = P\{A\}P\{D\},\ P\{BD\} = P\{B\}P\{D\}.$$

Because $B \quad A$, that

$$P\{(A \setminus B) \quad D\} = P\{A \quad D\} - P\{B \quad D\} = P\{A\}P\{D\} - P\{B\}P\{D\} =$$

$$= \ P\{A\} - P\{B\} \ P\{D\} = P\{A \setminus B\}P\{D\}.$$

Lemma 2 is proven.

Let us continue the proof of Theorem 2. It is easy to see that the independence of events A and B implies the independence of events $\overline{A} = \Omega \setminus A$ and B. Therefore, since ν is a random variable independent of the future, the event does not depend $(\nu > k-1) = \overline{(\nu \quad k-1)}$ on the σ -algebra . $F_{k,}$. Consequently, this event does not depend on σ the -algebra $\sigma(\xi_k)$. In exactly the same way, the event $(\nu > k-1)$ does not depend on σ the -algebra $\sigma(\xi_{k+1})$. Since the equality holds

$$(\nu = n) = (\nu > n-1) \setminus (\nu > n),$$

then, by Lemma 2, the event $(\nu = n)$ at $n \quad k$ does not depend on σ the -algebra $\sigma(\xi_k)$. Consequently, at $n \quad k$

$$E[g(\xi_k);\nu = n] = Eg(\xi_k)P\{\nu = n\}. \qquad (15)$$

Taking into account (15), we rewrite the right-hand side of equality (14) in the form

$$\sum_{k=1}\sum_{n=k}\frac{g(n)}{n}Eg(\xi_k)P\{\nu = n\} = \sum_{k=1}Eg(\xi_k)\sum_{n=k}\frac{g(n)}{n}P\{\nu = n\} =$$

$$= \sum_{n=1}\frac{1}{n}\sum_{k=1}^{n}Eg(\xi_k)\ \ g(n)P\{\nu = n\}. \qquad (16)$$

Now the proof of Theorem 2 follows from relations (13)-(16).
The proof of Theorem 2 is complete.

Remark 2. Theorem 2 is a generalization of B.A. Sevastyanov's result (Theorem 1). First, the condition of independence of the number of terms ν from the sequence of random variables (1) is replaced by a more general condition of independence of the random variable ν from the future. Second, if the random variables $g(\xi_k)$ have the same mathematical expectations, i.e.

$$Eg(\xi_1) = Eg(\xi_2) = ... = Eg(\xi_n) = ...,$$

then inequality (12) turns into estimate (4).

Remark 2. As shown by B.A. Sevastyanov [6], the conditions for the existence $Eg(\nu) <$ and membership of a function g in the class of convex functions G_2 cannot be weakened.

Remarks 4. Functions from the class G_1 with a constant $C = 1$ are called semimultiplicative . From the proof of Theorems 1-2, we can conclude that if the conditions for convexity of functions $g(\ )$ are replaced by the condition of concavity, then lower bounds for can be obtained $Eg(S_\nu)$. It should also be noted that semimultiplicative convex functions play an important role in the theory of linear operators (see [3], [4]).

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