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HIGHER-DEGREE EQUATIONS

G. Akhmedova

Annotation: This article outlines the key considerations for solving equations of degree higher than two and demonstrates solution methods through examples.

We recall some facts from the algebra course.

1. If $x = a$ is a root of the polynomial $P(x)$, then $P(x)$ is divisible by $x - a$ without a remainder.
2. Let all coefficients of the polynomial $P(x)$ be integers, with the leading coefficient equal to 1. If this equation has a rational root, then that root is an integer.
3. Let $P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$, with all coefficients being integers. If a root b of the polynomial is an integer, then b is a divisor of the constant term a_n.

The transformations applied when solving rational equations with integer coefficients result only in equivalent equations.

When solving fractional rational equations, multiplying both sides by the expression $Q(x)$ may introduce extraneous roots. Therefore, verification is necessary when solving fractional rational equations.

The primary methods for solving rational (and other) equations are:

1) Factoring;
2) Introducing a new variable.

The essence of the factoring method is as follows:

If $f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot \cdots \cdot f_n(x)$, then all solutions of the equation $f(x) = 0$ are the solutions of the system of equations $f_1(x) = 0; f_2(x) = 0; \ldots; f_n(x) = 0$. The converse is not true.

Example 1. Solve the equation:

$$x^3 + 4x^2 - 24 = 0$$

Solution. We use the trial-and-error method to solve this equation. According to the necessary condition for the existence of an integer root, we list the divisors of the constant term.

$$\alpha = \pm 1; \pm 2; \pm 3; \pm 4; \pm 6; \pm 8; \pm 12; \pm 24$$

Now we begin the trial.

Let's check the $\alpha = 1$, $1^3 + 4 \cdot 1^2 - 24 \neq 0$. $x = 1$ is not a root of the equation.

Let's check the $\alpha = -1$, $(-1)^3 + 4(-1)^2 - 24 \neq 0$. $x = -1$ is not a root of the equation.

Let's check the $\alpha = 2$, $2^3 + 4 \cdot 2^2 - 24 = 0$. $x = 2$ is a root of the equation.

The polynomial $x^3 + 4x^2 - 24$ is divisible by $x - 2$ without a remainder. Thus,

$x^3 + 4x^2 - 24 = (x - 2)(x^2 + 6x + 12)$.

Therefore, the given equation becomes $(x - 2)(x^2 + 6x + 12) = 0$.

From this, $x - 2 = 0$; $x^2 + 6x + 12 = 0$. The second equation has no real roots.

Answer: $x = 2$

Example 2. Solve the equation:

$$21x^3 + x^2 - 5x - 1 = 0$$

Solution. Equations with integer coefficients on the left-hand side and a constant term of 1 or -1 are reduced to the given equation by dividing term by term by the highest power of x. In this case, no roots are lost because $x = 0$ is not a root of an equation with a non-zero constant term. Then we replace $\frac{1}{x}$ with y.

In our example,

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$$21+\frac{1}{x}-\frac{5}{x^2}-\frac{1}{x^3}=0$$

is obtained. By setting $\frac{1}{x}=y$, we obtain the equation
$21+y-5y^2-y^3=0$.
This gives
$y^3+5y^2-y-21=0$.
Using the trial method as in Example 3, we find its integer root $y_1=-3$.
Dividing the polynomial y^3+5y^2-y-21 by $y+3$, we obtain the quadratic y^2+2y-7.
Its roots are $y_{2,3}=-1\pm2\sqrt{2}$. Since $x=\frac{1}{y}$,

$$x_1=-\frac{1}{3}, x_{2,3}=\frac{1\pm2\sqrt{2}}{7}$$

Answer: $x_1=-\frac{1}{3}, x_{2,3}=\frac{1\pm2\sqrt{2}}{7}$

Example 3. Solve the equation:

$$4x^3-10x^2+14x-5=0$$

Solution. To solve this example, we use a different transformation. We multiply both sides of the given equation by a number such that the coefficient of x^3 becomes the cube of some number. In our case, this number is 2. We multiply both sides of the equation by 2:

$$8x^3-20x^2+28x-10=0$$

Now we perform the substitution $y=2x$:
$y^3-5y^2+14y-10=0$.
Similar to the previous examples, using the trial method, we find that one root of the equation is $y_1=1$. This is the only root. From $x=\frac{y}{2}$, $x_1=\frac{1}{2}$ is the only root of the given equation.

Example 4. Solve the equation:

$$(x^2+x+4)^2+8x(x^2+x+4)+15x^2=0$$

Solution. We introduce the substitution $y=x^2+x+4$:
$y^2+8xy+15x^2=0$.
We solve this as a quadratic equation in y:

$$y_{1,2}=-4x\pm\sqrt{16x^2-15x^2}$$

Thus, $y_1=-3x, y_2=-5x$. Now we solve the system of equations $x^2+x+4=-3x$, $x^2+x+4=-5x$. This gives $x_{1,2}=-2, x_{3,4}=-3\pm\sqrt{5}$.

Example 5. Solve the equation:

$$x^2+\frac{9x^2}{(x+3)^2}=27$$

Solution. The left-hand side of the equation is a sum of squares. We add the expression $-2x\cdot\frac{3x}{x+3}$ to both sides of the equation and complete the square:

$$\left(x-\frac{3x}{x+3}\right)^2=27-6\frac{x^2}{x+3}$$

$$\left(\frac{x^2}{x+3}\right)^2+6\cdot\frac{x^2}{x+3}-27=0$$

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We introduce the substitution $y = \frac{x^2}{x+3}$ and obtain the equation
$y^2 + 6y - 27 = 0$.
This gives $y_1 =- 9, y_2 = 3$.
The given equation is reduced to solving the system of equations

$$\frac{x^2}{x+3} =- 9; \quad \frac{x^2}{x+3} = 3$$

The first equation has no roots. From the second equation, we get $x_{1,2} = \frac{3}{2} \pm \frac{3\sqrt{5}}{2}$. Both roots satisfy the condition $x + 3 \neq 0$. Therefore, the given equation has these roots.

Answer: $x_{1,2} = \frac{3}{2} \pm \frac{3\sqrt{5}}{2}$

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