

SOKHOTSKI–PLEMELJ FORMULAS AND THEIR ROLE IN THE THEORY OF SINGULAR INTEGRAL EQUATIONS

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Abstract: This paper discusses the Sokhotski–Plemelj formulas, which play a fundamental role in the regularization and solution of singular integral equations. The formulas describe the boundary behavior of Cauchy-type integrals and serve as the theoretical foundation for the Carleman–Mikhlin method and related analytical approaches. Through a synthesis of classical results and modern interpretations, this work presents the derivation, interpretation, and boundary properties of Cauchy-type integrals under Hölder continuity assumptions. The study highlights the analytical importance of these formulas in resolving discontinuities and establishing the existence and uniqueness of boundary values in complex analysis and mathematical physics.

Keywords: Singular integral equations, Sokhotski–Plemelj formulas, Cauchy-type integrals, regularization, boundary values, complex analysis.

Introduction

The study of singular integral equations is one of the key areas of modern mathematical analysis, with applications across elasticity theory, potential theory, aerodynamics, and electromagnetic field modeling. The Sokhotski–Plemelj formulas form the backbone of this theory, as they establish the relationship between the limiting values of Cauchy-type integrals on smooth or piecewise-smooth contours. The purpose of this research is to revisit these formulas, analyze their derivation, and demonstrate their application to the regularization and solution of boundary value problems.

Theoretical Background

Consider a function $\varphi(t)$ defined on a smooth contour L in the complex plane, satisfying the Hölder condition. The Cauchy-type integral is given by $F(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t-z} dt$, where the integral is understood in the sense of the Cauchy principal value. The Sokhotski–Plemelj formulas provide expressions for the boundary values of $F(z)$ as the point z approaches the contour from inside or outside. If we denote these limits by $F^+(x)$ and $F^-(x)$, respectively, then the formulas can be written as $F^+(x) = \frac{1}{2} \varphi(x) + \frac{1}{2\pi i} + \int_L \frac{\varphi(t)}{t-x} dt$,

$F^-(x) = -\frac{1}{2} \varphi(x) + \frac{1}{2\pi i} + \int_L \varphi(t)/(t-x) dt$. These relationships express the discontinuity of the Cauchy integral across the contour and are essential for transforming integral equations into Riemann–Hilbert boundary problems. If the contour has endpoints, the formulas can be modified by introducing the angle between the tangent vectors at the endpoints, preserving continuity in the limiting case.

Main Results and Discussion

The key theoretical implication of the Sokhotski–Plemelj formulas is that any Cauchy-type integral with a Hölder continuous density behaves as a double-layer potential, possessing well-defined one-sided limits on the contour. This property allows singular integral equations to be expressed as systems of boundary relations where the unknown function appears in both direct and principal value integrals. The resulting equations can then be treated using operator factorization or the Carleman–Mikhlin approach. The formulas also justify the existence and uniqueness of boundary values of analytic functions defined by singular integrals, providing the analytical link between potential theory and boundary value problems in complex analysis. Furthermore, for non-closed contours, appropriate extensions by auxiliary arcs ensure the validity of the formulas without loss of generality.

Conclusion

The Sokhotski–Plemelj formulas constitute a cornerstone of singular integral equation theory. They rigorously describe the discontinuity of Cauchy-type integrals across a contour and ensure the well-posedness of corresponding boundary value problems. By providing explicit relationships between the limiting values of analytic functions on a contour, these formulas not only serve as a theoretical tool for regularization but also enable practical analytical and numerical methods such as the Carleman–Mikhlin technique. Their universal nature makes them indispensable in mathematical physics, elasticity, and complex function theory.

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